Exercises on quantum singular value transformation

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1. Let U be a unitary partitioned into 2×2 matrix blocks of the same size, and Π be the orthogonal projection to the top-left block:

$$U = \begin{bmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{bmatrix}, \qquad \Pi = \begin{bmatrix} I & \\ & 0 \end{bmatrix}. \tag{1}$$

   (a) For any singular value decomposition $U_{00} = V_0 \Sigma W_0^\dagger$, prove that there exists unitaries V_1 and W_1 for which

$$\begin{aligned} U &= \begin{bmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{bmatrix} = \begin{bmatrix} V_0 & \\ & V_1 \end{bmatrix} \begin{bmatrix} \Sigma & \sqrt{I-\Sigma^2} \\ \sqrt{I-\Sigma^2} & -\Sigma \end{bmatrix} \begin{bmatrix} W_0^\dagger & \\ & W_1^\dagger \end{bmatrix}, \\ \Pi &= \begin{bmatrix} I & \\ & 0 \end{bmatrix} = \begin{bmatrix} V_0 & \\ & V_1 \end{bmatrix} \begin{bmatrix} I & \\ & 0 \end{bmatrix} \begin{bmatrix} V_0^\dagger & \\ & V_1^\dagger \end{bmatrix} = \begin{bmatrix} W_0 & \\ & W_1 \end{bmatrix} \begin{bmatrix} I & \\ & 0 \end{bmatrix} \begin{bmatrix} W_0^\dagger & \\ & W_1^\dagger \end{bmatrix}. \end{aligned} \tag{2}$$

   (b) Explain how to use this fact to analyze the operator sequence

$$e^{-i\phi_1(2\Pi - I)} U e^{-i\phi_2(2\Pi - I)} U^\dagger e^{-i\phi_3(2\Pi - I)} U e^{-i\phi_4(2\Pi - I)} U^\dagger \cdots \tag{3}$$

2. Let U be a unitary partitioned into 2×2 matrix blocks of the same size, and Π be the orthogonal projection to the top-left block:

$$U = \begin{bmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{bmatrix}, \qquad \Pi = \begin{bmatrix} I & \\ & 0 \end{bmatrix}. \tag{4}$$

   Additionally, assume that $U = U^\dagger$ is a Hermitian unitary and $\|U_{00}\| < 1$.

   (a) For any spectral decomposition $U_{00} = V_0 \Lambda V_0^\dagger$, prove that there exists a unitary V_1 for which

$$\begin{aligned} U &= \begin{bmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{bmatrix} = \begin{bmatrix} V_0 & \\ & V_1 \end{bmatrix} \begin{bmatrix} \Lambda & \sqrt{I-\Lambda^2} \\ \sqrt{I-\Lambda^2} & -\Lambda \end{bmatrix} \begin{bmatrix} V_0^\dagger & \\ & V_1^\dagger \end{bmatrix}, \\ \Pi &= \begin{bmatrix} I & \\ & 0 \end{bmatrix} = \begin{bmatrix} V_0 & \\ & V_1 \end{bmatrix} \begin{bmatrix} I & \\ & 0 \end{bmatrix} \begin{bmatrix} V_0^\dagger & \\ & V_1^\dagger \end{bmatrix}. \end{aligned} \tag{5}$$

   (b) Explain how to use this fact to analyze the operator sequence

$$e^{-i\phi_1(2\Pi - I)} U e^{-i\phi_2(2\Pi - I)} U e^{-i\phi_3(2\Pi - I)} U e^{-i\phi_4(2\Pi - I)} U \cdots \tag{6}$$

   (c) Find the eigenvalues and eigenvectors for

$$2\Pi - I, \qquad U, \qquad (2\Pi - I)U, \qquad U(2\Pi - I). \tag{7}$$

1. (a) **Solution:**

- Suppose that the given singular value decomposition $U_{00} = V_0 \Sigma W_0^\dagger$ is fixed. Construct an arbitrary singular value decomposition $U_{10} = V_1 \Sigma_1 W_1^\dagger$ for U_{10}. By the unitarity constraint,

$$I = U_{00}^\dagger U_{00} + U_{10}^\dagger U_{10} = W_0 \Sigma^2 W_0^\dagger + W_1 \Sigma_1^2 W_1^\dagger, \tag{8}$$

which implies

$$W_1 \Sigma_1^2 W_1^\dagger = W_0 (I - \Sigma^2) W_0^\dagger. \tag{9}$$

- Taking $\langle j| W_0^\dagger (\cdot) W_0 |j\rangle$ on both sides implies that $I - \Sigma^2 \geq 0$ is diagonal with nonnegative entries. This means that $W_0(I-\Sigma^2)W_0^\dagger$ and $W_1 \Sigma_1^2 W_1^\dagger$ are both singular value decompositions (spectral decompositions) of the same matrix.

- By permuting entries of Σ_1^2 and redefining W_1, we can say that $\Sigma_1^2 = I - \Sigma^2$ and

$$\Sigma_1 = \sqrt{I - \Sigma^2}. \tag{10}$$

- Without loss of generality, suppose that diagonal entries of $I - \Sigma^2$ are in nonincreasing order. Then the "uniqueness" of singular value decomposition guarantees that

$$W_1 = W_0 \begin{bmatrix} R_1 & & & & \\ & \ddots & & & \\ & & R_j & & \\ & & & \ddots & \\ & & & & R_d \end{bmatrix}, \tag{11}$$

where R_j are unitaries. However, we can then commute $R_j^\dagger$ through $\sqrt{I - \Sigma^2}$ and bundle it with V_1. Consequently, we have the singular value decomposition

$$U_{10} = V_1 \sqrt{I - \Sigma^2} W_0^\dagger. \tag{12}$$

- We now verify that $W_1 = U_{01}^\dagger V_0 \sqrt{I - \Sigma^2} - U_{11}^\dagger V_1 \Sigma$ is a unitary. Note that

$$W_1 = U_{01}^\dagger V_0 \sqrt{I - \Sigma^2} - U_{11}^\dagger V_1 \Sigma = \begin{bmatrix} U_{01}^\dagger & U_{11}^\dagger \end{bmatrix} \begin{bmatrix} V_0 & \\ & V_1 \end{bmatrix} \begin{bmatrix} \sqrt{I - \Sigma^2} \\ -\Sigma \end{bmatrix}. \tag{13}$$

This implies

$$\begin{aligned} W_1 W_1^\dagger &= \begin{bmatrix} U_{01}^\dagger & U_{11}^\dagger \end{bmatrix} \begin{bmatrix} V_0 & \\ & V_1 \end{bmatrix} \begin{bmatrix} I - \Sigma^2 & -\Sigma\sqrt{I - \Sigma^2} \\ -\Sigma\sqrt{I - \Sigma^2} & \Sigma^2 \end{bmatrix} \begin{bmatrix} V_0^\dagger & \\ & V_1^\dagger \end{bmatrix} \begin{bmatrix} U_{01} \\ U_{11} \end{bmatrix} \\ &= \begin{bmatrix} U_{01}^\dagger & U_{11}^\dagger \end{bmatrix} \begin{bmatrix} I - U_{00}U_{00}^\dagger & -U_{00}U_{10}^\dagger \\ -U_{10}U_{00}^\dagger & I - U_{10}U_{10}^\dagger \end{bmatrix} \begin{bmatrix} U_{01} \\ U_{11} \end{bmatrix} \\ &= \begin{bmatrix} U_{01}^\dagger & U_{11}^\dagger \end{bmatrix} \begin{bmatrix} I & \\ & I \end{bmatrix} \begin{bmatrix} U_{01} \\ U_{11} \end{bmatrix} - \begin{bmatrix} U_{01}^\dagger & U_{11}^\dagger \end{bmatrix} \begin{bmatrix} U_{00} \\ U_{10} \end{bmatrix} \begin{bmatrix} U_{00}^\dagger & U_{10}^\dagger \end{bmatrix} \begin{bmatrix} U_{01} \\ U_{11} \end{bmatrix} \\ &= I - 0 = I. \end{aligned} \tag{14}$$

- Finally, we verify that

$$\begin{aligned}
V_0\sqrt{I-\Sigma^2}W_1^\dagger &= V_0\sqrt{I-\Sigma^2}\left(\sqrt{I-\Sigma^2}V_0^\dagger U_{01}-\Sigma V_1^\dagger U_{11}\right)\\
&=(I-U_{00}U_{00}^\dagger)U_{01}-U_{00}U_{10}^\dagger U_{11}=U_{01},\\
-V_1\Sigma W_1^\dagger &= -V_1\Sigma\left(\sqrt{I-\Sigma^2}V_0^\dagger U_{01}-\Sigma V_1^\dagger U_{11}\right)\\
&=-U_{10}U_{00}^\dagger U_{01}+(I-U_{10}U_{10}^\dagger)U_{11}=U_{11}.
\end{aligned} \tag{15}$$

(b) **Solution:**
Omitted.

2. (a) **Solution:**

- Suppose that the given spectral decomposition $U_{00}=V_0\Lambda V_0^\dagger$ is fixed. Proceed as in the proof of generic CS decomposition, we can construct the singular value decomposition
$$U_{10}=V_1\sqrt{I-\Lambda^2}V_0^\dagger. \tag{16}$$

- The Hermicity assumption implies
$$U_{01}=V_0\sqrt{I-\Lambda^2}V_1^\dagger. \tag{17}$$
Consequently,
$$U=\begin{bmatrix}V_0 & \\ & V_1\end{bmatrix}\begin{bmatrix}\Lambda & \sqrt{I-\Lambda^2}\\ \sqrt{I-\Lambda^2} & V_1^\dagger U_{11}V_1\end{bmatrix}\begin{bmatrix}V_0^\dagger & \\ & V_1^\dagger\end{bmatrix}. \tag{18}$$

- By the assumption $U^2=I$, off-diagonal blocks of U^2 are zero, which implies
$$\Lambda\sqrt{I-\Lambda^2}+\sqrt{I-\Lambda^2}V_1^\dagger U_{11}V_1=0. \tag{19}$$

- Since $\|U_{00}\|<1$, $\|\Lambda\|<1$ so $\sqrt{I-\Lambda^2}$ is diagonal and invertible. This means
$$V_1^\dagger U_{11}V_1=-\Lambda. \tag{20}$$

(b) **Solution:**
Omitted.

(c) **Solution:**
See arXiv:2410.18178, Table 4.